Dressed electrostatic solitary waves in quantum dusty pair plasmas

M. Akbari-Moghanjoughi  
Department of Physics, Faculty of Sciences, Azarbaijan University of Tarbiat Moallem, 51745-406 Tabriz, Iran

(Received 11 January 2010; accepted 23 March 2010; published online 6 May 2010)

Quantum-hydrodynamics model is applied to investigate the nonlinear propagation of electrostatic solitary excitations in a quantum dusty pair plasma. A Korteweg de Vries evolution equation is obtained using reductive perturbation technique and the higher-nonlinearity effects are derived by solving the linear inhomogeneous differential equation analytically using Kodama–Tanahashi renormalizing method. The possibility of propagation of bright- and dark-type solitary excitations is examined. It is shown that a critical value of quantum diffraction parameter $H$ exists, on either side of which, only one type of solitary propagation is possible. It is also found that unlike for the first-order amplitude component, the variation of $H$ parameter dominantly affects the soliton amplitude in higher-order approximation. The effect of fractional quantum number density on compressive and rarefactive soliton dynamics is also discussed. © 2010 American Institute of Physics. [doi:10.1063/1.3392289]

I. INTRODUCTION

Currently, quantum plasma is becoming one of attracting fields of plasma research to scientists both from theoretical and experimental points of view. This is because of its broad applicability in micro- and nanomanufactures. Quantum electronic transport effects play a crucial role in metallic semiconductor nanostructured materials, such as nanoparticles, quantum wells, quantum wires, and quantum dots, where the electrons and holes are in a degenerate state and obey the Pauli exclusion principle. Hence, in such a dense state of matter, the classical statistical models break down and quantum mechanical consideration should be taken into account. Quantum effects are also one of the main ingredients of nonlinear processes in ultradense plasma environments such as white dwarfs, active galactic nuclei, neutron stars, and many other dense astrophysical environments.

Recently, a semiclassical approach has been applied to investigate the nonlinear dynamics of solitary structures in an electron-positron-ion plasma, where the electrons and positrons are treated as an ideal Fermi gas and the ions are treated as classical fluid (due to the fact that in a dense quantum plasma the ion Fermi speed is much smaller than that of the electron’s). For instance, Dubinov et al. have developed a nonlinear theory of ion-acoustic (IA) waves in ideal plasma with degenerate electrons and classical ions. In ordinary plasma there are criteria for applicability of such models, since, the quantum degeneracy effects start playing an effective role when the de Broglie thermal wavelength $\lambda_B=(2\pi\hbar/k_BT)^{1/2}$ of electrons is equal or larger than the average interelectron distances, $n^{-1/3}$ (Ref. 1), or equivalently, when the temperature $T$ is comparable or lower than the electron Fermi temperature $T_F=eF_e/k_B$, where $eF_e$ is the electron Fermi energy. However, this requirement is well satisfied for metallic compounds, semiconductors, and nanostructured materials.

On the other hand, a more realistic model of quantum plasma is quantum hydrodynamics (QHD) model which incorporates the quantum statistical pressure as well as the quantum-force tunneling effect for degenerate plasma ingredients. The QHD model alike the classical hydrodynamics includes the ion (electron/positron) continuity, momentum, and Poisson equations. However, unlike classical fluids, quantum plasma, instead of dissipation, exhibits dispersion caused by the quantum tunneling effects associated with the Bohm potential. More recently, QHD model has been extended to explain the propagations of IA solitary excitations in unmagnetized electron-ion (e-i) plasma. Chatterjee et al. have investigated the propagation of dressed IA solitons in dusty pair plasma. However, dressed electrostatic (ES) solitons has not been studied in the framework of QHD model to our current knowledge.

In this study, the QHD model is applied to three component pair plasma to investigate the higher-nonlinearity effects on ES solitary excitations. The organization of the article is as follows. Description of QHD state equations is given in Sec. II. Reductive perturbation method is applied and the Korteweg de Vries (KdV) evolution equation is obtained in Sec. III. The stationary solution to higher-order soliton amplitude approximation is given in Sec. IV. Section V presents the discussions based on the numerical analysis and, finally, Sec. VI devotes to the concluding remarks.

II. DESCRIPTION OF QUANTUM PLASMA STATE

A three-component plasma system containing pair of positive and negative particles (fullerene or electron-positron pair) and heavy inertial particles (say dusts) may be described as a fermionic degenerate-gas governed by a one-dimensional continuum-pressure continuity equation set. The QHD-model for such plasma environment resembles the classical counterpart, except in this case the quantum pressure which arises from spin one-half integer Fermi statistics...
and tunneling potential, i.e., Bohm potential (see Ref. 15), due to wavelike nature of particles, is taken into account. However, the pressure of dust particulates is ignored in forthcoming algebra.

The dimensional form of the closed set of basic equations in this model can be written as

$$\frac{\partial n_+}{\partial t} + \frac{\partial u_+ n_+}{\partial x} = 0,$$

$$\frac{\partial n_-}{\partial t} + \frac{\partial u_- n_-}{\partial x} = 0,$$

$$\frac{\partial u_+}{\partial t} + u_+ \frac{\partial u_+}{\partial x} = \frac{Z_p}{m_+} \frac{\partial \phi}{\partial x} - \frac{\partial p_+}{\partial x} + \frac{k^2}{2m_p^2} \frac{\partial}{\partial x} \left[ \frac{1}{n_+} \frac{\partial^2 \sqrt{n_+}}{\partial x^2} \right] + \frac{\hbar^2}{2m_p^2} \frac{\partial}{\partial x} \left[ \frac{1}{n_+} \frac{\partial^2 \sqrt{n_+}}{\partial x^2} \right],$$

$$\frac{\partial u_-}{\partial t} + u_- \frac{\partial u_-}{\partial x} = \frac{Z_p}{m_-} \frac{\partial \phi}{\partial x} - \frac{\partial p_-}{\partial x} + \frac{k^2}{2m_p^2} \frac{\partial}{\partial x} \left[ \frac{1}{n_-} \frac{\partial^2 \sqrt{n_-}}{\partial x^2} \right] + \frac{\hbar^2}{2m_p^2} \frac{\partial}{\partial x} \left[ \frac{1}{n_-} \frac{\partial^2 \sqrt{n_-}}{\partial x^2} \right].$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\epsilon_0} \left[ Z_p(n_- - n_+) - Z_d N_d \right].$$

where the ± sign is used to differentiate between charges and h is the normalized Plank constant. Also, subscripts p and d refer to pair and dust, respectively. In forthcoming we assume $Z_p=1$ for simplicity. In a one-dimensional dense and degenerate plasma environment the quantum pressure of fermions in zero-temperature assumption is governed by Pauli exclusion principle, which relates to particle quantum number density through the following relations:

$$p_\pm = \frac{m_\pm v_{F\pm}^2 n_{\pm,0}^3}{3n_{\pm,0}^2}, \quad v_{F\pm} = \sqrt{\frac{2E_{F\pm}}{m_\pm}}, \quad E_{F\pm} = k_B T_{F\pm},$$

where quantities $v_{F\pm}$, $E_{F\pm}$, and $T_{F\pm}$ denote Fermi velocity, Fermi energy, and Fermi temperature, respectively and $n_{\pm,0}$ is pair particles equilibrium number densities. Also, using standard definitions, it is remarked that in a one-dimensional degenerate Fermi gas the equilibrium quantum number density is related to Fermi temperature, i.e.,

$$T_{F\pm} = \frac{(3\pi^2\hbar n_{\pm,0}^3)^{2/3}}{2m_p k_B}.$$  

It is also remarked that in zero-temperature Fermi gas, the collisions are limited due to a process called Fermi blocking; hence, such systems may be considered as collisionless. This is fairly valid for metals in which the Fermi temperature is well above room temperature. In a fully degenerate plasma regime, the de Broglie thermal wavelength $\lambda_B = (2m_p k_B T)^{1/2}$ of particles is equal to or larger than the average interparticle distance $n^{-1/3}$.

To obtain a dimensionless set of state equations, we use the following scalings:

$$x \rightarrow C_n x, \quad t \rightarrow t, \quad n \rightarrow n_n, \quad n \rightarrow n_n .$$

$$u \rightarrow u C_n, \quad \phi \rightarrow 2k_B T_{F\pm} / e.$$  

where $\omega_p = \sqrt{e^2 n_{\pm,0} / e_d m_p}$ and $C_n = \sqrt{2k_B T_{F\pm} / m_p}$ are the characteristic plasma frequency and sound speed, respectively. The normalized compact set of plasma equations, assuming equal mass and charge ($Z=1$) for pair species, read as

$$\frac{\partial n_\alpha}{\partial t} + u_\alpha \frac{\partial n_\alpha}{\partial x} = 0,$$

$$\frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} = -S_\alpha \frac{\partial \phi}{\partial x} - n_\alpha^2 \frac{\partial^2 n_\alpha}{\partial x^2} + \frac{H^2}{2} \frac{\partial}{\partial x} \left[ \frac{1}{n_\alpha} \frac{\partial^2 \sqrt{n_\alpha}}{\partial x^2} \right],$$

$$\frac{\partial^2 \phi}{\partial x^2} = - \sum_\alpha S_\alpha n_\alpha - Z_d N_d.$$  

The $\alpha = \{+, -\}$ index denotes the particle chargetype and $S_\alpha$ can have the values of $S_n=\{+1, -1\}$ for the respective charges. The quantities $u_\alpha$, $n_\alpha$, and $\phi$ refer to the velocity, density of $\alpha$-charged particles and the ES potential, respectively. Also, a normalized parameter $H = \hbar \omega_p / 2k_B T_{F\pm}$ known as quantum diffraction parameter (the ratio of particle plasma energy to the Fermi energy) is introduced. One may argue that in the $H \rightarrow 0$ limit the classical case is achieved, however, this is not true since, in fact $H \approx 1 / h$ [Eq. (3)]. Besides, in QHD model, we have used the quantum pressure (number density) instead of classical one.

The quasineutrality condition is defined by Poisson’s relation at thermodynamic equilibrium state of the last equation in Eq. (5) as

$$n_{\pm,0} - n_{\pm,0} - Z_d N_d = 0,$$

or equivalently

$$\beta + \delta = 1, \quad \beta = \frac{n_{\pm,0}}{n_{\pm,0}}, \quad \delta = \frac{Z_d N_d}{n_{\pm,0}},$$

where $Z_d$ and $N_d$ are the atomic number and density of positive background heavy ions (dusts).

The dispersion of linear waves around homogeneous equilibrium is given by the Fourier analysis of the reduced equation set [Eq. (5)] as

$$\frac{1}{1 - \frac{\omega^2}{k^2} + \frac{H^2 k^2}{4}} + \frac{\beta}{2 - \frac{\omega^2}{k^2} + \frac{H^2 k^2}{4}} \cdot -k^2.$$  

The dispersion plots for the optical and acoustic-ES linear waves are given in Fig. 1. The long-wavelength limit linear-dispersion relations are given as
\[
\omega^2 = (1 + \beta) + (1 - \beta + \beta^2)k^2 + \left[ \frac{H^2}{4} + \frac{\beta(1 - \beta - \beta^2)}{(1 + \beta)} \right] k^4.
\]

However, QHD model does not apply to the very small wavelength limit (Ref. 20). At the particular case of \( \beta = 1, \ H = 0 \), which corresponds to classical pair plasma with no dust impurities, the dispersion of the form \( \omega \approx k \) is obtained and for the case of plasma with negative thermal-ions and inertial positive dusts with equal concentrations \( (\beta = 0, \ H = 0) \), the dispersion relation reduces only to an optical branch, \( \omega \approx \sqrt{1 + k^2} \), where the acoustic branch disappears. These limiting cases are in complete agreement with the classical analogs given in Ref. 21. However, the mentioned limits are excluded from our discussion for reasons which will follow next.

### III. REDUCTIVE PERTURBATION AND KDV EVOLUTION EQUATION

Suppose a wavelike perturbation moving with a phase speed \( \lambda \) in the following stretched-coordinate

\[
\xi = \varepsilon^{1/2} \left( x - \lambda \tau \right),
\]

\[
\tau = \varepsilon^{3/2} \tau.
\]

The corresponding plasma state equations are introduced into the new coordinate in the following form:

\[
\varepsilon \frac{\partial n_a}{\partial \tau} - \lambda \left( \frac{\partial n_a}{\partial \xi} + \frac{\partial u_a}{\partial \xi} \right) = 0,
\]

\[
\varepsilon n_a \frac{\partial u_a}{\partial \tau} - \lambda n_a \frac{\partial u_a}{\partial \xi} + n_a^3 \frac{\partial u_a}{\partial \xi} = - S_a n_a^2 \frac{\partial \phi}{\partial \xi} + \frac{H^2}{4} \left[ \frac{n_a^2}{\partial \xi} \right]^3 - 2 n_a \frac{n_a^2}{\partial \xi} \frac{\partial n_a}{\partial \xi} + n_a^2 \frac{\partial n_a}{\partial \xi}.
\]

where the parameter \( \varepsilon \) describes the nonlinearity strength to be very small, positive and real number. Asymptotic expansion of plasma variables away from thermodynamics equilibrium can be done using following orderings:

\[
n_a = n_{a,0} + \varepsilon n_{a,1} + \varepsilon^2 n_{a,2} + \ldots,
\]

\[
u_a = \nu_{a,0} + \varepsilon \nu_{a,1} + \varepsilon^2 \nu_{a,2} + \ldots,
\]

\[
\phi = \phi_{1,0} + \varepsilon \phi_{1,1} + \varepsilon^2 \phi_{1,2} + \ldots
\]

By applying the orderings [Eq. (12)] in Eq. (11) and isolating distinct perturbation-orders, in the leading-order we obtain the following relations between plasma quantities

\[
\lambda \frac{\partial n_{a,0}^{(1)}}{\partial \xi} + n_{a,0} \frac{\partial \nu_{a,0}^{(1)}}{\partial \xi} = 0,
\]

\[
\lambda \frac{\partial \nu_{a,0}^{(1)}}{\partial \xi} + S \frac{\partial \phi_{1,0}^{(1)}}{\partial \xi} + n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi} = 0,
\]

\[
- \sum_a S \frac{\partial n_{a,0}^{(1)}}{\partial \xi} = 0,
\]

from which we deduce the first-order perturbed components as

\[
u_{a,0}^{(1)} = U_{1a} \phi^{(1)}, \quad U_{1a} = \frac{\lambda S_a}{\lambda^2 - n_{a,0}^2},
\]

\[
n_{a,0}^{(1)} = N_{1a} \phi^{(1)}, \quad N_{1a} = \frac{n_{a,0} S_a}{\lambda^2 - n_{a,0}^2},
\]

Within this approximation the nonlinear dispersion relation is also obtained, which in compact form read as

\[
\sum_a n_{a,0} \frac{n_{a,0}}{\lambda^2 - n_{a,0}^2} = 0.
\]

From compatibility requirement we can also calculate the phase speed of waves in terms of fractional particles density

\[
\lambda^2 = \beta.
\]

In the next higher-order by following the same route, we get

\[
\frac{\partial n_{a,0}^{(2)}}{\partial \tau} - (\lambda + n_{a,0} + \lambda n_{a,0}) \frac{\partial n_{a,0}^{(2)}}{\partial \xi} + n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi} = 0,
\]

\[
n_{a,0}^{(2)} \frac{\partial \nu_{a,0}^{(1)}}{\partial \tau} - n_{a,0} \frac{\partial \nu_{a,0}^{(1)}}{\partial \xi} = - 3 \lambda n_{a,0}^2 n_{a,0} \frac{\partial u_{a,0}^{(1)}}{\partial \xi} + n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi} + 3 \lambda n_{a,0}^2 n_{a,0} \frac{\partial \phi_{1,0}^{(1)}}{\partial \xi} + n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi}
\]

\[
+ 3 S_a n_{a,0}^2 n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi} + S_a n_{a,0}^2 \frac{\partial \phi_{1,0}^{(1)}}{\partial \xi} + 4 n_{a,0} \frac{\partial n_{a,0}^{(1)}}{\partial \xi}
\]

\[
+ n_{a,0} \frac{\partial n_{a,0}^{(2)}}{\partial \xi} = 0.
\]
\[
\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = - \sum \alpha S_{\alpha} \rho_{\alpha}^{(2)}. \tag{17c}
\]

By making use of dispersion relation to eliminate the common terms from Eq. (17), we obtain the KdV evolution equation which describes the first-order nonlinear evolution of the wave amplitude
\[
\frac{\partial \phi^{(1)}}{\partial \tau} + AB \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{A}{2} \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0. \tag{18}
\]

The coefficients of KdV equation read as
\[
A = \frac{4 \beta (1 - \beta)^2 - H^2 (1 + \beta)}{4 (1 + \beta) \sqrt{\beta}}, \tag{19}
\]
\[
B = \frac{3 + \beta (2 + 3 \beta)}{2A (1 - \beta^2) \sqrt{\beta}}. \tag{20}
\]

The second-order density and velocity perturbation components are derived using Eqs. (14) and (17) as
\[
n_a^{(2)} = N_{1a} \phi^{(2)} + N_{12a} \phi^{(1)} + N_{13a} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}, \tag{21}
\]

and
\[
\frac{\partial^2 \phi^{(2)}}{\partial \xi^2} = - \sum \alpha S_{\alpha} \rho_{\alpha}^{(3)}. \tag{25c}
\]

Equation (25) together with Eqs. (14), (18), (23), and (24) give rise to the following linear inhomogeneous equation, the solution of which along with Eq. (18), describes the higher-order wave amplitude evolution
\[
\frac{\partial \phi^{(2)}}{\partial \tau} + AB \frac{\partial \phi^{(1)} \phi^{(2)}}{\partial \xi} + \frac{A}{2} \frac{\partial^3 \phi^{(2)}}{\partial \xi^3} = L_0 \left[ L_1 \frac{\partial^2 \phi^{(1)}}{\partial \tau^2} + L_2 \frac{\partial \phi^{(1)}}{\partial \tau} \frac{\partial \phi^{(1)}}{\partial \xi} + L_3 \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \right]. \tag{26}
\]

The coefficients of linear inhomogeneous equation are given in the Appendix.

\[
\frac{\partial u_a^{(2)}}{\partial \tau} = U_{1a} \phi^{(2)} + U_{12a} \phi^{(1)} + U_{13a} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}, \tag{22}
\]

where
\[
N_{12a} = \frac{1}{2} \left[ 3 S_a N_{1a} + 4 n_{a,0} N_{2a} - 2 \lambda A N_{1a} \right], \tag{23}
\]
\[
N_{13a} = - \frac{N_{1a}}{4} \left[ 4 \lambda A + H^2 \right], \tag{24}
\]

and
\[
U_{12a} = \frac{N_{12a}}{n_{a,0}} - \frac{N_{1a} U_{1a}}{n_{a,0}} + \frac{A B N_{1a} n_{a,0}}{(\lambda^2 - n_{a,0}^2)}, \tag{25a}
\]

In the next higher-order of \(\varepsilon\) we obtain the following relations:
\[
\frac{\partial n_a^{(2)}}{\partial \tau} - \lambda n_{a,0} \frac{\partial n_a^{(3)}}{\partial \xi} + n_{a,0} \frac{\partial u_a^{(2)}}{\partial \xi} + \frac{\partial u_a^{(1)} n_a^{(2)}}{\partial \xi} + \frac{\partial u_a^{(2)} n_a^{(1)}}{\partial \xi} = 0, \tag{25b}
\]

IV. CALCULATION OF HIGHER-ORDER PERTURBATION AMPLITUDE

In this section, we present the stationary solution of Eqs. (18) and (26) bounded with the appropriate boundary conditions, which describe the wave amplitude up to the second-order using the renormalization technique. We may rewrite Eq. (26) in simpler form as
\[
\frac{\partial \phi^{(2)}}{\partial \tau} + AB \frac{\partial \phi^{(1)} \phi^{(2)}}{\partial \xi} + \frac{A}{2} \frac{\partial^3 \phi^{(2)}}{\partial \xi^3} = \Gamma(\phi^{(1)}), \tag{27}
\]

where
\[
\Gamma(\phi^{(1)}) = L_0 \left[ L_1 \frac{\partial^2 \phi^{(1)}}{\partial \tau^2} + L_2 \frac{\partial \phi^{(1)} \phi^{(1)}}{\partial \xi} + L_3 \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \right]. \tag{28}
\]
Following the methodology introduced by Kodama and Taniuti (Ref. 22), we write

\[
\frac{\partial \bar{\phi}^{(1)}}{\partial \tau} + AB \frac{\partial \bar{\phi}^{(1)}}{\partial \xi} + \frac{A}{2} \frac{\partial \bar{\phi}^{(1)}}{\partial \xi} + \delta \nu \frac{\partial \bar{\phi}^{(1)}}{\partial \xi} = 0, \tag{29a}
\]

\[
\frac{\partial \bar{\phi}^{(2)}}{\partial \tau} + AB \frac{\partial \bar{\phi}^{(2)}}{\partial \xi} + \frac{A}{2} \frac{\partial \bar{\phi}^{(2)}}{\partial \xi} + \delta \nu \frac{\partial \bar{\phi}^{(2)}}{\partial \xi} = \Gamma(\bar{\phi}^{(1)}) + \delta \nu \frac{\partial \bar{\phi}^{(1)}}{\partial \xi}. \tag{29b}
\]

Therefore, in the moving frame \(\zeta = \xi - (\nu + \delta \nu) \tau\) Eq. (29) become

\[
\frac{\partial^2 \bar{\phi}^{(1)}}{\partial \zeta^2} + \left( B \bar{\phi}^{(1)} - \frac{2 \nu}{A} \right) \bar{\phi}^{(1)} = 0, \tag{30a}
\]

\[
\int_{-\infty}^{\xi} \left[ \Gamma(\bar{\phi}^{(1)}) d\zeta + \delta \nu \bar{\phi}^{(1)} \right] = \frac{L_0 \bar{\phi}_0}{3A^5} \left[ 360L_5 + A^2 \bar{\phi}_0 (12AL_1 - 6L_2 + 12L_3) + (L_4 - 2ABL_1) A^2 \bar{\phi}_0 \right] \text{sech}^4 \left( \frac{\xi}{\Lambda} \right)
- \frac{L_0 \bar{\phi}_0}{\Lambda^4} [120L_5 - 2A^2 \bar{\phi}_0 (L_2 + L_3) + 2AL_1 A^2 \bar{\phi}_0] \text{sech}^4 \left( \frac{\xi}{\Lambda} \right) + \bar{\phi}_0 \left( \frac{16L_0 L_5}{\Lambda^4} + \delta \nu \right) \text{sech}^2 \left( \frac{\xi}{\Lambda} \right). \tag{33}
\]

By setting the last term in Eq. (33) to zero we eliminate the resonant term in Eq. (30) which simplifies it to the following expression:

\[
\frac{\partial^2 \bar{\phi}^{(2)}}{\partial \zeta^2} + 2 \left( B \bar{\phi} \text{sech}^2 \left( \frac{\xi}{\Lambda} \right) - \frac{\nu}{A} \right) \bar{\phi}^{(2)} = \frac{2L_0 \bar{\phi}_0}{3A^4 \Lambda^4} \left[ 360L_5 + A^2 \bar{\phi}_0 (12AL_1 - 6L_2 + 12L_3) + (L_4 - 2ABL_1) A^2 \bar{\phi}_0 \right] \text{sech}^4 \left( \frac{\xi}{\Lambda} \right)
- \frac{2L_0 \bar{\phi}_0}{\Lambda^4} [120L_5 - 2A^2 \bar{\phi}_0 (L_2 + L_3) + 2AL_1 A^2 \bar{\phi}_0] \text{sech}^4 \left( \frac{\xi}{\Lambda} \right). \tag{34}
\]

Once again, by changing the variable \(\eta = \tanh(\xi/\Lambda)\), Eq. (34) is transformed into the familiar form of associated Legendre function

\[
\frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial}{\partial \eta} \right] \bar{\phi}^{(2)} + 3(3 + 1) - \frac{4}{(1 - \eta^2)} \bar{\phi}^{(2)} = \Psi(\eta), \tag{35}
\]

The particular solutions of Eq. (35) are of the form

\[
\Phi_{2p}(\eta) = \psi_1(\eta) P_3^2(\eta) + \psi_2(\eta) Q_3^2(\eta), \tag{37}
\]

where

\[
\Psi(\eta) = \chi_1 (1 - \eta^2)^2 + \chi_2 (1 - \eta^2), \quad P_3^2(\eta) = 15 \eta (1 - \eta^2), \quad Q_3^2(\eta) = \frac{15}{2} \eta (1 - \eta^2) \ln \left( \frac{1 + \eta}{1 - \eta} \right) + \frac{2}{1 - \eta^2} - 15(1 - \eta^2) + 5,
\]

\[
\chi_1 = \frac{2L_0 \bar{\phi}_0}{3A^4 \Lambda^4} \left[ 360L_5 + A^2 \bar{\phi}_0 [12AL_1 - 6L_2 + 12L_3]ight.
+ (L_4 - 2ABL_1) A^2 \bar{\phi}_0], \tag{36}
\]

\[
\chi_2 = \frac{2L_0 \bar{\phi}_0}{A^4 \Lambda^4} [120L_5 - 2A^2 \bar{\phi}_0 (L_2 + L_3) + 2AL_1 A^2 \bar{\phi}_0].
\]
By making use of Eqs. (36)–(39), we can simplify the particular solution to

$$\Phi_2(\eta) = (1 - \eta^2) \left[ \frac{\chi_2}{6} + \frac{\chi_1}{4} - \frac{\chi_3}{8} (1 - \eta^2) \right].$$

(40)

The complementary solution of Eq. (35) is

$$\Phi_{2c}(\eta) = C_1 \Phi_2^1(\eta) + C_2 \Phi_2^2(\eta),$$

(41)

in which the first part can be eliminated by renormalizing of the wave amplitude. The coefficient $C_2$ of the second part, on the other hand, is vanished due to the boundary conditions [Eq. (31)]. Therefore, considering only the particular solution, the stationary solution for the combination of the first- and the second-order amplitude perturbations is of the form

$$\Phi(\xi) = \bar{\Phi}^{(1)} + \bar{\Phi}^{(2)} = \text{sech}^2 \left( \frac{\xi}{\Lambda} \right) \left[ \phi_0 + \left[ \frac{\chi_2}{6} + \frac{\chi_1}{4} - \frac{\chi_3}{8} \text{sech}^2 \left( \frac{\xi}{\Lambda} \right) \right] \right].$$

(42)

In writing Eq. (42), notice that the $e$ and $e^2$ coefficients have been absorbed.

V. NUMERICAL ANALYSIS

Figure 1 shows the dispersion curves of acoustic- and optical-ES solitary excitation for fixed value of relative dust concentration and different values of quantum diffraction parameter $H$. As it is clearly observed, the increase in $H$ value leads to higher dispersion for both acoustic and optical branches.

Regarding the possibility and types of ES solitary structure some comments have to be made. As it is concluded from the coefficients of KdV equation [Eqs. (19) and (20)], the product $AB$, which is inversely proportional to soliton amplitude, is always positive, but the coefficient $A$ may be negative or positive depending on the values of $H$ and $\beta$. This indicates that the solitons width vanishes at some critical $H$ values given by

$$H_{cr} = 2(1 - \beta) \sqrt{\frac{\beta}{1 + \beta}},$$

(43)

below (above) which $A$ is positive (negative) and the solitary structures must be compressive (rarefactive). This is similar to the case of quantum LA solitary waves treated in Ref. 17, where a critical value of $H_{cr} = 2$ has been obtained. Figure 2(a) clearly distinguishes two separate branches shown with solid and dashed curves for each given value of fractional dust concentration, $\delta$, separated by a critical value $H_{cr}$, which correspond to bright- and dark-type solitary excitations, respectively. The maximum value of the critical quantum diffraction parameter occurs at $H_{cr,max} = 0.67$ for $\beta_{max} = 0.28$, which is well below that of metallic electrons ($6 \approx H_{metallic} \approx 2$). However, the corresponding value of $H$ can be much smaller for heavy ions such as fullerene pair. The variation of first-order soliton-amplitude is depicted in Fig. 2(b), which shows a maximum value around $\beta = 0.30$.

One notes that, the width of solitary structure [Eq. (32)] vanishes at $H = H_{cr}$, $\beta = 0$ or $\beta = 1$. The later case corresponds to the quantum symmetric pair plasma, $n_{-q} = n_{q,0}$, which is analogous to the classical counterpart (Ref. 21), where there is no KdV-type solitary propagation. However, the former is a purely quantum mechanical case. It is also noted from

FIG. 2. (Color online) (a) The bright-type (solid curve, $\nu = 0.2$) and dark-type (dashed curve, $\nu = -0.2$) soliton branches connected at critical value, $H_{cr}$. and (b) the variation of first-order soliton-amplitude approximation with respect to $\beta$ for different values of phase speeds, $\nu$. Different dash sizes refer to and are related appropriately to different values of $v$ in (a).
Eq. (43) that the cases of \( \beta=0 \) and \( \beta=1 \) coincide with the case, \( H_{1,2}=0 \) (negligible quantum diffraction).

Furthermore, as it is clear from Figs. 3(d)–3(f) and Figs. 4(d)–4(f), the first-order amplitude approximation, determined by the product \( AB \), is independent from the value of \( H \). This is, within the first-order approximation, in agreement with findings for quantum 1A solitons presented in Ref. 17. However, from Figs. 3(a)–3(c) and Figs. 4(a)–4(c), it is observed that in the higher-order approximation the soliton amplitude is strongly dependent to the value of quantum diffraction parameter. Figures 3 and 4 also indicate that the first-order approximations lead to the bell-shaped structure for the solitary excitations, whereas the higher-order corrections reveal that the solitary structures are of \( W \)-shape, in general. These \( W \)-shaped solitary structures have also been reported for classical dressed ES solitons given in Ref. 21.

It is further remarked that for bright-type solitary excitations (Fig. 3) the decrease in the fractional dust concentration \( \delta \) leads to a decrease in the width of the soliton (in both first- and second-approximation orders), whereas, in the case of dark-type solitary structures (Fig. 4) the width (in both first- and second-approximation orders) is nearly unaffected by the change in the value of \( \delta \) (dust-to-negative ion number-density ratio). In the first-order amplitude approximation, Figs. 3 and 4 show that the quantum diffraction effect has also a significant effect on the soliton width, so that the increase in the value of the quantum diffraction parameter, \( H \), gives rise to decrease (increase) in the width of the bright-type (dark-type) solitons. On the other hand, Figs. 3(a)–3(c) and Figs. 4(a)–4(c) reveal that a small change in the magnitude of quantum diffraction parameter, unlike the first-order approximation, drastically alters the shape of solitary structures in higher-order approximation. In fact for bright-type (dark-type) solitary structures the increase in the quantum diffraction effect gives rise to the increase (decrease) in the higher-order amplitude component, for a fixed value of rela-

**FIG. 3.** (Color online) Profiles of solitary excitation structures for bright-type soliton is given for first-order [(a)–(c)] and second-order [(a)–(c)] potential approximations. The variation of soliton width and amplitude is also shown for a fixed value of phase speed, \( \nu=0.1 \), different values of positive-to-negative particle number-density ratios, \( \beta \) (within each column) and different values of quantum diffraction parameter, \( H \) (within each plot). Different dash sizes refer to and are related appropriately to different values of \( H \).

**FIG. 4.** (Color online) Profiles of solitary excitation structures for dark-type soliton is given for first-order [(d)–(f)] and second-order [(a)–(c)] potential approximations. The variation of soliton width and amplitude is also shown for a fixed value of phase speed, \( \nu=-0.1 \), different values of positive-to-negative particle number-density ratios, \( \beta \) (within each column) and different values of quantum diffraction parameter, \( H \) (within each plot). Different dash sizes refer to and are related appropriately to different values of \( H \).
tive dust concentration [Figs. 3(a)–3(c) and 4(a)–4(c)]. However, the behavior of changes in the higher-order amplitude of both bright- and dark-type solitons due to variations of relative dust concentration, $\delta$, is rather irregular.

VI. CONCLUSION

We applied the standard reductive perturbation method to describe the nonlinear propagation of ES solitary propagation in QHD-model which accounts for the quantum effects. The KdV evolution equation is obtained from lower approximations and the higher-order amplitude approximation is derived by solving the linear inhomogeneous equation by using the renormalization technique. It was shown that dark- and bright-type soliton regions are separated by a critical value of quantum diffraction parameter, $H_{\text{cr}}$, in either side of which only single type of soliton can occur. It is also found that, unlike for the first-order amplitude, the quantum diffraction parameter significantly affects the wave amplitude in higher-order approximations.

APPENDIX: COEFFICIENTS OF LINEAR INHOMOGENEOUS EQUATION

$$L_0 = \left( \sum_{\alpha} 2n_{\alpha,0}^2 \frac{\lambda N_{1,\alpha}}{\lambda^2 - n_{\alpha,0}^2} \right)^{-1},$$

$$L_1 = - \sum_{\alpha} S_{\alpha} \frac{2n_{\alpha,0} U_{12,\alpha} + \frac{3}{2} U_{1,\alpha} N_{1,\alpha}}{\lambda^2 - n_{\alpha,0}^2},$$

$$L_2 = - \sum_{\alpha} S_{\alpha} \frac{N_{13,\alpha} (3 + 4n_{\alpha,0} N_{1,\alpha}) - H^2}{2} \left( \frac{N_{1,\alpha}^2}{n_{\alpha,0}} + 3N_{12,\alpha} \right) + \sum_{\alpha} S_{\alpha} \frac{3AB(n_{\alpha,0} U_{1,\alpha} - \lambda S_{\alpha} N_{13,\alpha}) - \lambda(U_{13,\alpha} N_{1,\alpha} + N_{13,\alpha} U_{1,\alpha})}{\lambda^2 - n_{\alpha,0}^2},$$

$$L_3 = - \sum_{\alpha} S_{\alpha} \frac{4n_{\alpha,0} N_{1,\alpha} N_{13,\alpha} - AB(n_{\alpha,0} U_{13,\alpha} + \lambda N_{13,\alpha})}{\lambda^2 - n_{\alpha,0}^2} + \sum_{\alpha} S_{\alpha} \frac{U_{13,\alpha} (3\lambda N_{1,\alpha} + n_{\alpha,0} U_{1,\alpha}) + H^2}{2} \left( \frac{N_{12,\alpha} + \frac{N_{1,\alpha}^2}{n_{\alpha,0}}}{2} - \lambda(U_{13,\alpha} N_{1,\alpha} + N_{13,\alpha} U_{1,\alpha}) \right) + \sum_{\alpha} S_{\alpha} \frac{3N_{12,\alpha} U_{1,\alpha} - 2U_{12,\alpha} N_{1,\alpha} - 3\lambda N_{1,\alpha} U_{1,\alpha} + 2n_{\alpha,0} U_{12,\alpha} U_{1,\alpha} + 3U_{1,\alpha}^2 N_{1,\alpha} + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2 + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2 + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2}{\lambda^2 - n_{\alpha,0}^2},$$

$$L_4 = - \sum_{\alpha} S_{\alpha} \frac{N_{1,\alpha}^2}{n_{\alpha,0}} + 3S_{\alpha} N_{12,\alpha} + 2N_{1,\alpha} (4n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2 + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2 + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2 + 2n_{\alpha,0} N_{12,\alpha} + 3N_{1,\alpha}^2}{\lambda^2 - n_{\alpha,0}^2},$$

$$L_5 = \sum_{\alpha} S_{\alpha} \frac{2\lambda n_{\alpha,0} U_{13,\alpha} + H^2 N_{13,\alpha} + 2\lambda N_{13,\alpha}}{\lambda^2 - n_{\alpha,0}^2}. $$